

# A HILBERT-TYPE THEOREM FOR SPACELIKE SURFACES WITH CONSTANT GAUSSIAN CURVATURE IN $\mathbb{H}^2 \times \mathbb{R}_1$

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**ABSTRACT.** There are examples of complete spacelike surfaces in the Lorentzian product  $\mathbb{H}^2 \times \mathbb{R}_1$  with constant Gaussian curvature  $K \leq -1$ . In this paper, we show that there exists no complete spacelike surface in  $\mathbb{H}^2 \times \mathbb{R}_1$  with constant Gaussian curvature  $K > -1$ .

## 1. INTRODUCTION

In 1900 Liebmann [10] characterized the spheres as the unique complete surfaces with constant positive Gaussian curvature in  $\mathbb{R}^3$ . One year later, in 1901 Hilbert [8] showed that it does not exist any complete surface with constant negative Gaussian curvature in  $\mathbb{R}^3$ . Finally, every complete surface with zero Gaussian curvature in  $\mathbb{R}^3$  must be a straight cylinder over a complete, planar and simple curve, as was proved independently by Hartman and Nirenberg in 1958 [7], Stoker in 1961 [13] and Massey in 1962 [11]. The Liebmann and Hilbert theorems are easily extended to complete surfaces in  $\mathbb{S}^3$  and  $\mathbb{H}^3$ , since their proofs depend basically on the Codazzi equation, which is the same in any space form. In 2007 Aledo, Espinar and Gálvez [4] extended the Liebmann and Hilbert theorems to the case of complete surfaces with constant Gaussian curvature in the Riemannian homogeneous product spaces  $\mathbb{S}^2 \times \mathbb{R}$  and  $\mathbb{H}^2 \times \mathbb{R}$ . Specifically, they showed that the only complete surfaces with constant Gaussian curvature  $K > 0$  in  $\mathbb{H}^2 \times \mathbb{R}$  (resp.  $K > 1$  in  $\mathbb{S}^2 \times \mathbb{R}$ ) are rotational surfaces. In addition, they proved the non existence of complete surfaces with constant Gaussian curvature  $K < -1$  in  $\mathbb{H}^2 \times \mathbb{R}$  and  $\mathbb{S}^2 \times \mathbb{R}$ .

Recently, in [2] the authors jointly with Aledo complemented the results in [4] by showing that the slices are the only compact two-sided surfaces in  $\mathbb{S}^2 \times \mathbb{R}$  whose angle function does not change sign and have constant Gaussian curvature. Moreover, a similar result is valid for spacelike complete surfaces in the Lorentzian product space

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*Date:* May 2009.

2000 *Mathematics Subject Classification.* 53C42, 53C50.

The authors are partially supported by MEC project MTM2007-64504, and Fundación Séneca project 04540/GERM/06, Spain. This research is a result of the activity developed within the framework of the Programme in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Regional Agency for Science and Technology (Regional Plan for Science and Technology 2007-2010).

$\mathbb{S}^2 \times \mathbb{R}_1$ : the only complete spacelike surfaces in the Lorentzian product  $\mathbb{S}^2 \times \mathbb{R}_1$  with constant Gaussian curvature are the slices [2, Corollary 9]. However, in the proof of these results we use as a main tool the compactness of  $\mathbb{S}^2$ , so it can not be extended to surfaces in the Lorentzian product  $\mathbb{H}^2 \times \mathbb{R}_1$ . Actually, slices  $\mathbb{H}^2 \times \{t_0\}$ ,  $t_0 \in \mathbb{R}$ , are trivial examples of complete spacelike surfaces in  $\mathbb{H}^2 \times \mathbb{R}_1$  with constant Gaussian curvature  $K = -1$ . On the other hand, in [2, Example 12] we have recently given an example of a non trivial complete entire spacelike graph in  $\mathbb{H}^2 \times \mathbb{R}_1$  with constant Gaussian curvature  $K$  for every value of  $K$  such that  $K < -1$ . Therefore, it seems a natural question to study the existence or non existence of complete spacelike surfaces in  $\mathbb{H}^2 \times \mathbb{R}_1$  with constant Gaussian curvature  $K > -1$ . In this context, the following non existence result is proved:

**Theorem 1.** *There exists no complete spacelike surface in  $\mathbb{H}^2 \times \mathbb{R}_1$  with constant Gaussian curvature  $K > -1$ .*

The proof of Theorem 1 for  $K > 0$  is a consequence of the Bonnet-Myers theorem taking into account that there is no compact surface in  $\mathbb{H}^2 \times \mathbb{R}_1$  (see Section 3). On the other hand, in the case  $-1 < K \leq 0$  the proof follows the ideas introduced in [4, Theorem 3] and it is based on two geometric tools: the abstract theory of Codazzi pairs and the construction of a new complete metric on the surface obtained when we deform the induced metric in the direction of the height function. However, in difference with the proof of [4, Theorem 3], our proof of Theorem 1 only requires tensorial computations.

In Section 2 we introduce the necessary notions about spacelike surfaces in  $\mathbb{H}^2 \times \mathbb{R}_1$  as well as the notion of a Codazzi pair and the theorem of Wissler, which is fundamental in the proof of Theorem 1. The complete proof of Theorem 1 is given in Section 3. Finally, in the Appendix we compare the geometry of a spacelike surface in  $\mathbb{H}^2 \times \mathbb{R}_1$  with the geometry of the same surface endowed with the Riemannian metric obtained by deformation of the induced metric by a fixed function.

*Note added in proof.* After submission of this paper, we were informed by Gálvez, Jiménez and Mira that our Theorem 1 can be seen also as an application of their general correspondence results between isometric immersions in [6] and the non existence result of complete surfaces with constant Gaussian curvature  $K < -1$  in the Riemannian product  $\mathbb{H}^2 \times \mathbb{R}$ .

## 2. PRELIMINARIES

### 2.1. Spacelike surfaces in $\mathbb{H}^2 \times \mathbb{R}_1$ .

Let  $(\mathbb{H}^2, g_{\mathbb{H}^2})$  be the hyperbolic plane, and let us consider the product manifold  $\mathbb{H}^2 \times \mathbb{R}$  endowed with the Lorentzian metric

$$g = \pi_{\mathbb{H}}^*(g_{\mathbb{H}^2}) - \pi_{\mathbb{R}}^*(dt^2),$$

where  $\pi_{\mathbb{H}}$  and  $\pi_{\mathbb{R}}$  denote the projections from  $\mathbb{H}^2 \times \mathbb{R}$  onto each factor. For simplicity, we will write

$$g = g_{\mathbb{H}^2} - dt^2,$$

and we will denote by  $\mathbb{H}^2 \times \mathbb{R}_1$  the 3-dimensional product manifold  $\mathbb{H}^2 \times \mathbb{R}$  endowed with that Lorentzian metric.

A smooth immersion  $f : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}_1$  of a connected surface  $\Sigma^2$  is said to be a spacelike surface if  $f$  induces a Riemannian metric on  $\Sigma$ , which as usual is also denoted by  $g$ . It is interesting to remark that in that case, since

$$\partial_t = (\partial/\partial t)_{(x,t)}, \quad x \in \mathbb{H}^2, t \in \mathbb{R},$$

is a unitary timelike vector field globally defined on the ambient spacetime  $\mathbb{H}^2 \times \mathbb{R}_1$ , there exists a unique unitary timelike normal field  $N$  globally defined on  $\Sigma$  which is in the same time-orientation as  $\partial_t$ . That is,

$$g(N, \partial_t) \leq -1 < 0 \quad \text{on } \Sigma.$$

We will refer to  $N$  as the future-pointing Gauss map of  $\Sigma$ , and we will denote by  $\Theta : \Sigma \rightarrow (-\infty, -1]$  the smooth function on  $\Sigma$  given by  $\Theta = g(N, \partial_t)$ . The function  $\Theta$  measures the hyperbolic angle  $\theta$  between the future-pointing vector fields  $N$  and  $\partial_t$  along  $\Sigma$ . Indeed, they are related by  $\cosh \theta = -\Theta$ .

In order to fix notation, let  $\bar{\nabla}$  and  $\nabla$  denote the Levi-Civita connections in  $\mathbb{H}^2 \times \mathbb{R}_1$  and  $\Sigma$ , respectively. Then the Gauss and Weingarten formulae for the spacelike surface  $f : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}_1$  are given by

$$(1) \quad \bar{\nabla}_X Y = \nabla_X Y - g(AX, Y)N$$

and

$$(2) \quad AX = -\bar{\nabla}_X N,$$

for any tangent vector fields  $X, Y \in T\Sigma$ . Here  $A : T\Sigma \rightarrow T\Sigma$  stands for the shape operator (or second fundamental form) of  $\Sigma$  with respect to its future-pointing Gauss map  $N$ . As is well known, the Gaussian curvature  $K$  of the surface  $\Sigma$  is described in terms of  $A$  and the curvature of the ambient spacetime by the Gauss equation, which is given by

$$(3) \quad K = \bar{K} - \det A,$$

where  $\bar{K}$  denotes the sectional curvature in  $\mathbb{H}^2 \times \mathbb{R}_1$  of the plane tangent to  $\Sigma$ . It is not difficult to see that the Gauss equation (3) can be written as

$$(4) \quad K = -\Theta^2 - \det A.$$

On the other hand, let  $\bar{R}$  denote the curvature tensor of  $\mathbb{H}^2 \times \mathbb{R}_1$ . The Codazzi equation of the spacelike surface  $\Sigma$  describes the tangent component of  $\bar{R}(X, Y)N$ ,

for any tangent vector fields  $X, Y \in T\Sigma$ , in terms of the derivative of the shape operator and it is given by

$$(5) \quad (\bar{R}(X, Y)N)^\top = (\nabla_X A)Y - (\nabla_Y A)X,$$

where  $\nabla_X A$  denotes the covariant derivative of  $A$ , that is,

$$(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y).$$

From now on, if  $Z$  is a vector field along the immersion  $f : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}_1$ , then  $Z^\top \in T\Sigma$  stands for the tangential component of  $Z$  along  $\Sigma$ , that is,  $Z = Z^\top - g(N, Z)N$ . It can be seen that, as the hyperbolic plane is a complete surface of constant Gaussian curvature  $-1$ ,  $\bar{R}$  can be simplified and the Codazzi equation (5) becomes

$$(6) \quad (\nabla_X A)Y = (\nabla_Y A)X - \Theta(g(X, \partial_t^\top)Y - g(Y, \partial_t^\top)X),$$

(for the details on the above computations see, for instance, [1, 3]).

Given a spacelike surface  $f : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}_1$ , the height function of  $\Sigma$ , denoted by  $h$ , is defined as the projection of  $\Sigma$  onto  $\mathbb{R}$ , that is,  $h \in \mathcal{C}^\infty(\Sigma)$  is the smooth function given by  $h = \pi_{\mathbb{R}} \circ f$ . Observe that the gradient of  $\pi_{\mathbb{R}}$  on  $\mathbb{H}^2 \times \mathbb{R}_1$  is

$$\bar{\nabla}\pi_{\mathbb{R}} = -g(\bar{\nabla}\pi_{\mathbb{R}}, \partial_t)\partial_t.$$

Therefore, the gradient of  $h$  on  $\Sigma$  is

$$\nabla h = (\bar{\nabla}\pi_{\mathbb{R}})^\top = -\partial_t^\top.$$

Since  $\partial_t^\top = \partial_t + \Theta N$ , we easily get

$$(7) \quad \|\nabla h\|^2 = \Theta^2 - 1,$$

where  $\|\cdot\|$  denotes the norm of a vector field on  $\Sigma$ . Since  $\partial_t$  is parallel on  $\mathbb{H}^2 \times \mathbb{R}_1$  we have that

$$(8) \quad \bar{\nabla}_X \partial_t = 0$$

for any tangent vector field  $X \in T\Sigma$ . Writing  $\partial_t = -\nabla h - \Theta N$  along the surface  $\Sigma$  and using Gauss (1) and Weingarten (2) formulae, we easily get from (8) that

$$(9) \quad \nabla_X \nabla h = \Theta AX$$

for every  $X \in T\Sigma$ .

## 2.2. Codazzi pairs.

An important geometrical tool for the proof of our result is the abstract theory of Codazzi pairs following [12]. Let  $(A, B)$  be a pair of real quadratic forms on a 2-dimensional surface  $\Sigma$  such that  $A$  is a Riemannian metric. Associated to this pair it is possible to define its extrinsic curvature in an abstract way as the quotient

$$(10) \quad K(A, B) = \frac{\det B}{\det A}.$$

On the other hand, since  $A$  is a Riemannian metric, it has associated a Levi-Civita connection  $\nabla^A$ , a Riemann curvature tensor  $R_A$  defined, as usual, by

$$R_A(X, Y)Z = \nabla_{[X, Y]}^A Z - [\nabla_X^A, \nabla_Y^A]Z$$

for any  $X, Y, Z \in T\Sigma$  and the corresponding Gaussian curvature

$$(11) \quad K_A = \frac{A(R_A(X, Y)X, Y)}{Q_A(X, Y)},$$

being  $Q_A(X, Y) = A(X, X)A(Y, Y) - A(X, Y)^2$  for any  $X, Y \in T\Sigma$ .

The pair  $(A, B)$  is said to be a Codazzi pair if it satisfies the Codazzi equation of a space form, that is,

$$(12) \quad (\nabla_X^A S)(Y) - (\nabla_Y^A S)(X) = 0$$

for every  $X, Y \in T\Sigma$ ,  $S : T\Sigma \rightarrow T\Sigma$  being the endomorphism in  $T\Sigma$   $A$ -metrically equivalent to  $B$ , that is

$$B(X, Y) = A(SX, Y),$$

and  $(\nabla_X^A S)$  the covariant derivative of  $S$ ,

$$(\nabla_X^A S)(Y) = \nabla_X^A(SY) - S(\nabla_X^A Y).$$

The following result, due to Wissler, will be fundamental in the proof of our result:

**Theorem 2** ([14], [15]). *Let  $(A, B)$  be a Codazzi pair with constant negative extrinsic curvature  $K(A, B)$ . Then, if  $A$  is complete  $\inf_{\Sigma} |K_A| = 0$ .*

### 3. PROOF OF THEOREM 1

Let us recall first that any complete spacelike surface  $f : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}_1$  is necessarily diffeomorphic to  $\mathbb{H}^2$ . Actually, it is not difficult to see that  $\Pi = \pi_M \circ f : \Sigma \rightarrow \mathbb{H}^2$  satisfies  $\Pi^*(g_{\mathbb{H}^2}) \geq g$ . Therefore,  $\Pi$  is a local diffeomorphism which increases the distance between the Riemannian surfaces  $(\Sigma, g)$  and  $(\mathbb{H}^2, g_{\mathbb{H}^2})$ . The completeness of  $\Sigma$  implies that  $\Pi$  is a covering map [9, Chapter VIII, Lemma 8.1]. Moreover, since  $\mathbb{H}^2$  is simply connected,  $\Pi$  is a global diffeomorphism. As a direct consequence of it, there exists no compact spacelike surface in  $\mathbb{H}^2 \times \mathbb{R}_1$ . On the other hand, from the Bonnet-Myers theorem any Riemannian surface with positive constant Gaussian curvature is necessarily compact. Consequently, there exists no complete spacelike surface in  $\mathbb{H}^2 \times \mathbb{R}_1$  with positive constant Gaussian curvature.

Let us assume now that  $f : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}_1$  is a complete spacelike surface with constant Gaussian curvature  $-1 < K \leq 0$ , and let us consider the Riemannian metric on  $\Sigma$  defined by

$$(13) \quad \tilde{g} = g + c dh^2 \geq g,$$

where  $c$  is the positive constant  $c = \frac{1}{K+1} > 0$ . Since  $g$  is a complete metric by assumption and  $\tilde{g} \geq g$ ,  $\tilde{g}$  is also a complete metric on  $\Sigma$ .

Let  $\alpha : T\Sigma \times T\Sigma \rightarrow \mathbb{R}$  denote the second fundamental form of the surface  $f : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}_1$ , that is,  $\alpha(X, Y) = g(AX, Y)$ .

**Claim.** *We assert that  $(\tilde{g}, \alpha)$  is a Codazzi pair with constant negative extrinsic curvature*

$$K(\tilde{g}, \alpha) = -(K + 1) < 0.$$

To prove this claim, observe first that the endomorphism  $\tilde{A} : T\Sigma \rightarrow T\Sigma$  which is  $\tilde{g}$ -metrically equivalent to  $\alpha$  can be written in terms of  $A$ . In fact for any  $X, Y \in T\Sigma$  it holds

$$g(AX, Y) = \alpha(X, Y) = \tilde{g}(\tilde{A}X, Y),$$

and from (13)

$$g(AX, Y) = \tilde{g}(AX, Y) - cAX(h)Y(h).$$

Therefore we get

$$(14) \quad \tilde{A}X = AX - cg(AX, \nabla h)\tilde{\nabla}h,$$

for any  $X \in T\Sigma$ . On the other hand, by the definition of the gradient of a function, and by the expression (13) for the metric  $\tilde{g}$ , it yields

$$X(h) = \tilde{g}(\tilde{\nabla}h, X) = g(\nabla h, X) = \tilde{g}(\nabla h, X) - c\|\nabla h\|^2\tilde{g}(\tilde{\nabla}h, X),$$

for any  $X \in T\Sigma$ . Then,

$$\tilde{\nabla}h = \frac{1}{1 + c\|\nabla h\|^2}\nabla h,$$

so (14) becomes

$$(15) \quad \tilde{A}X = AX - \frac{c}{1 + c\|\nabla h\|^2}g(AX, \nabla h)\nabla h.$$

It is also possible to express the Levi-Civita connection of the metric  $\tilde{g}$ ,  $\tilde{\nabla}$ , in terms of the differential operators related to the metric  $g$ , obtaining the relation

$$(16) \quad \tilde{\nabla}_X Y = \nabla_X Y + \frac{c}{1 + c\|\nabla h\|^2}\nabla^2 h(X, Y)\nabla h$$

for any  $X, Y \in T\Sigma$ ,  $\nabla^2$  being the Hessian operator of the surface  $f : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}_1$ , (see the Appendix for the details).

From (16) and (15) we get with a straightforward computation that

$$(17) \quad \begin{aligned} (\tilde{\nabla}_Y \tilde{A})X &= \tilde{\nabla}_Y(\tilde{A}X) - \tilde{A}(\tilde{\nabla}_Y X) \\ &= (\nabla_Y A)X - \frac{c}{1 + c\|\nabla h\|^2}g((\nabla_Y A)X, \nabla h)\nabla h \\ &\quad - \frac{c}{1 + c\|\nabla h\|^2}g(AX, \nabla h)\nabla_Y \nabla h + T(X, Y), \end{aligned}$$

where  $T$  is the symmetric  $(0, 2)$  tensor on  $\Sigma$  given by

$$T(X, Y) = \frac{c^2}{(1 + c\|\nabla h\|^2)^2} \Theta(g(AY, \nabla h)g(AX, \nabla h) + g(AX, Y)g(A(\nabla h), \nabla h)) \nabla h \\ - \frac{c}{1 + c\|\nabla h\|^2} \nabla^2 h(X, Y) A(\nabla h).$$

Using the Codazzi equation (6), we observe that

$$g((\nabla_Y A)X - (\nabla_X A)Y, \nabla h) = 0.$$

Therefore, using again the Codazzi equation (6) and the expression (9), we obtain from (17) that

$$(18) \quad (\tilde{\nabla}_Y \tilde{A})X - (\tilde{\nabla}_X \tilde{A})Y = \Theta(g(Y, \nabla h)X - g(X, \nabla h)Y) \\ - \Theta \frac{c}{1 + c\|\nabla h\|^2} (g(AX, \nabla h)AY - g(AY, \nabla h)AX).$$

To check that the left hand side of (18) vanishes, it is enough to proof that it vanishes when we consider as vector fields  $\{E_1, E_2\}$  a local  $g$ -orthonormal frame of  $T\Sigma$  which diagonalizes the shape operator. It is worth pointing out that such a frame does not always exist; problems can occur when the multiplicity of the principal curvatures changes and also at the points where the principal curvatures are not differentiable. However, we can consider the open dense subset of  $\Sigma$ ,  $\Sigma'$ , consisting of points at which the number of distinct principal curvatures is locally constant. Then, for every  $p \in \Sigma'$  there exists a local  $g$ -orthonormal frame defined on a neighbourhood of  $p$  that diagonalizes  $A$ , that is,  $\{E_1, E_2\}$  such that  $AE_1 = \lambda_1 E_1$  and  $AE_2 = \lambda_2 E_2$  with each  $\lambda_i$  smooth, see, for instance, [5, Paragraph 16.10]. We will work on  $\Sigma'$ , and the conclusion will be valid in all the surface  $\Sigma$  by a continuity argument. Considering these vector fields, (18) becomes

$$(\tilde{\nabla}_{E_2} \tilde{A})E_1 - (\tilde{\nabla}_{E_1} \tilde{A})E_2 = \Theta \left( 1 + \lambda_1 \lambda_2 \frac{c}{1 + c\|\nabla h\|^2} \right) (g(E_2, \nabla h)E_1 - g(E_1, \nabla h)E_2),$$

which vanishes, since using the Gauss equation (4) and the relation (7) we get

$$\lambda_1 \lambda_2 \frac{c}{1 + c\|\nabla h\|^2} = -(K + \Theta^2) \frac{\frac{1}{K+1}}{1 + \frac{1}{K+1}\|\nabla h\|^2} = -\frac{K + \Theta^2}{K + 1 + \|\nabla h\|^2} = -1.$$

It remains to compute the extrinsic curvature of the Codazzi pair  $(\tilde{g}, \alpha)$ . Let  $\{E_1, E_2\}$  be a local  $g$ -orthonormal frame of  $T\Sigma$ , then

$$\tilde{g}(E_i, E_i) = 1 + cg(E_i, \nabla h)^2 \quad \text{and} \quad \tilde{g}(E_1, E_2) = cg(E_1, \nabla h)g(E_2, \nabla h).$$

Therefore, we have

$$\det \tilde{g} = (1 + cg(E_1, \nabla h)^2)(1 + cg(E_2, \nabla h)^2) - c^2 g(E_1, \nabla h)^2 g(E_2, \nabla h)^2 = 1 + c\|\nabla h\|^2,$$

so using the equations (4) and (7), the extrinsic curvature of  $(\tilde{g}, \alpha)$  is given by

$$K(\tilde{g}, \alpha) = \frac{\det \alpha}{\det \tilde{g}} = \frac{\det A}{1 + c\|\nabla h\|^2} = \frac{-(K+1)(K+\Theta^2)}{K+1+\|\nabla h\|^2} = -(K+1) < 0.$$

This finishes the proof of our Claim.

Consider now  $\Sigma'' \subset \Sigma$  the subset in  $\Sigma$  where the height function  $h$  is non constant.  $\Sigma''$  is an open dense subset of  $\Sigma$ , since in other case it would exist an open subset  $\Omega \subset \Sigma$  where  $h|_{\Omega}$  is constant. Then, from expressions (7) and (9)  $\Theta|_{\Omega} = -1$  and  $A|_{\Omega} = 0$ . Therefore, from the Gauss equation (4) it would be  $K = -1$ , which contradicts our assumption. By Lemma 3 in the Appendix, the Gaussian curvature of the surface  $(\Sigma, \tilde{g})$ ,  $\tilde{K}$ , can be written in terms of the Gaussian curvature of the surface  $(\Sigma, g)$  as

$$(19) \quad \tilde{K} = \frac{K(1 + c\|\nabla h\|^2) + c \det \nabla^2 h}{(1 + c\|\nabla h\|^2)^2}$$

in  $\Sigma''$ . And by continuity (19) holds in  $\Sigma$ . Observe that from the expressions (9) and (8) and from the Gauss equation (4) we get

$$(20) \quad \det \nabla^2 h = \Theta^2 \det A = -\Theta^2(K + \Theta^2) = -(1 + \|\nabla h\|^2)(K + 1 + \|\nabla h\|^2).$$

Therefore, (19) becomes

$$(21) \quad \tilde{K} = \frac{(1 - c)K - c(1 + \|\nabla h\|^2)^2}{(1 + c\|\nabla h\|^2)^2}.$$

If we consider in (21)  $\tilde{K}$  as a function of  $\|\nabla h\|^2$ , then  $\tilde{K}$  is a monotonous decreasing function. Therefore, evaluating it at 0 and using that  $c = 1/(K+1)$  we have

$$(22) \quad \inf \tilde{K} \leq \sup \tilde{K} = (1 - c)K - c = K - 1 < 0.$$

Summing up, we have proven that  $(\tilde{g}, \alpha)$  is a Codazzi pair with negative constant extrinsic curvature,  $\tilde{g}$  being a complete Riemannian metric with Gaussian curvature  $\tilde{K}$  verifying (22), which contradicts the theorem of Wissler, Theorem 2. Therefore, it can not exist any complete spacelike surface  $f : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}_1$  with constant Gaussian curvature  $-1 < K \leq 0$ , as we were assuming, which completes the proof of Theorem 1.

#### APPENDIX: RELATING THE GEOMETRY OF $(\Sigma, g)$ AND $(\Sigma, \tilde{g})$ .

Given a Riemannian surface  $(\Sigma, g)$ , a non constant smooth function  $u \in \mathcal{C}^\infty(\Sigma)$  and a positive constant  $c > 0$ , it makes sense to consider the new Riemannian surface  $(\Sigma, \tilde{g})$ , where

$$(23) \quad \tilde{g} = g + cdu^2 \geq g.$$

Therefore  $(\Sigma, \tilde{g})$  is obtained by deformation of the metric  $g$  in the direction of the function  $u$ . Observe that in the particular case where  $\Sigma$  is a spacelike surface in



$\mathbb{H}^2 \times \mathbb{R}_1$  and  $u$  is the height function of  $\Sigma$ , the situation is the one presented in Section 3. Our aim in this appendix is to obtain some relations between the geometry of  $(\Sigma, g)$  and  $(\Sigma, \tilde{g})$ , giving general versions of the expressions (16) and (21).

We begin by studying the relation between the Levi-Civita connections of  $(\Sigma, \tilde{g})$ ,  $\tilde{\nabla}$ , and  $(\Sigma, g)$ ,  $\nabla$ . Using the Koszul formula and the expression (23) for  $\tilde{g}$  we have

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X Y, Z) &= X(\tilde{g}(Y, Z)) + Y(\tilde{g}(Z, X)) - Z(\tilde{g}(X, Y)) \\ &\quad - \tilde{g}(X, [Y, Z]) + \tilde{g}(Y, [Z, X]) + \tilde{g}(Z, [X, Y]) \\ &= 2g(\nabla_X Y, Z) + c[X(Y(u)Z(u)) + Y(Z(u)X(u)) - Z(X(u)Y(u)) \\ &\quad - X(u)(YZ - ZY)(u) + Y(u)(ZX - XZ)(u) \\ &\quad + Z(u)(XY - YX)(u)] \\ &= 2g(\nabla_X Y, Z) + 2cX(Y(u))Z(u), \end{aligned}$$

for any  $X, Y, Z \in T\Sigma$ . On the other hand, from (23) we get

$$\tilde{g}(\tilde{\nabla}_X Y, Z) = g(\tilde{\nabla}_X Y, Z) + c\tilde{\nabla}_X Y(u)Z(u),$$

so we obtain

$$(24) \quad \tilde{\nabla}_X Y = \nabla_X Y - c \left( \tilde{\nabla}_X Y(u) - X(Y(u)) \right) \nabla u$$

for any  $X, Y \in T\Sigma$ . It follows from here that

$$\tilde{\nabla}_X Y(u) = \nabla_X Y(u) - c\tilde{\nabla}_X Y(u)\|\nabla u\|^2 + cX(Y(u))\|\nabla u\|^2.$$

Therefore, we have

$$(25) \quad \tilde{\nabla}_X Y(u) = \frac{1}{1 + c\|\nabla u\|^2} (\nabla_X Y(u) + cX(Y(u))\|\nabla u\|^2).$$

Finally, substituting (25) into (24) we get

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{c}{1 + c\|\nabla u\|^2} (\nabla_X Y(u) - X(Y(u))) \nabla u$$

for any  $X, Y \in T\Sigma$ . Or equivalently,

$$(26) \quad \tilde{\nabla}_X Y = \nabla_X Y + \frac{c}{1 + c\|\nabla u\|^2} \nabla^2 u(X, Y) \nabla u,$$

$\nabla^2$  being the Hessian operator of the surface  $(\Sigma, g)$ .

In the following lemma, we obtain the relation between the Gaussian curvature  $\tilde{K}$  of  $(\Sigma, \tilde{g})$  and the Gaussian curvature  $K$  of  $(\Sigma, g)$ .

**Lemma 3.** *Let  $(\Sigma, g)$  be a Riemannian surface,  $u \in \mathcal{C}^\infty(\Sigma)$  a non constant smooth function and  $c > 0$  a positive constant. Then, the Gaussian curvature  $\tilde{K}$  of the*

Riemannian surface  $(\Sigma, \tilde{g} = g + cdu^2)$  is given by

$$\tilde{K} = \frac{K(1 + c\|\nabla u\|^2) + c \det \nabla^2 u}{(1 + c\|\nabla u\|^2)^2},$$

where  $K$ ,  $\nabla$  and  $\nabla^2$  denote the Gaussian curvature, the gradient and the Hessian operator of  $(\Sigma, g)$ , respectively.

*Proof.* Let  $\{E_1, E_2\}$  be a local  $g$ -orthonormal frame on  $T\Sigma$  such that  $E_2 \perp \nabla u$ . Then,

$$(27) \quad K = g(R(E_1, E_2)E_1, E_2),$$

and

$$(28) \quad \tilde{K} = \frac{\tilde{g}(\tilde{R}(E_1, E_2)E_1, E_2)}{\tilde{Q}(E_1, E_2)},$$

where  $\tilde{Q}(E_1, E_2) = \tilde{g}(E_1, E_1)\tilde{g}(E_2, E_2) - \tilde{g}(E_1, E_2)^2 = 1 + c\|\nabla u\|^2$ , and  $R$  and  $\tilde{R}$  stand for the Riemann curvature tensors of  $(\Sigma, g)$  and  $(\Sigma, \tilde{g})$ , respectively. Therefore we need the relation between  $\tilde{R}$  and  $R$ . Since

$$\tilde{R}(E_1, E_2)E_1 = \tilde{\nabla}_{[E_1, E_2]}E_1 - [\tilde{\nabla}_{E_1}, \tilde{\nabla}_{E_2}]E_1,$$

we will study each term separately. From the expression (26), we have

$$(29) \quad \begin{aligned} \tilde{\nabla}_{\tilde{\nabla}_{E_1}E_2}E_1 &= \tilde{\nabla}_{\nabla_{E_1}E_2}E_1 + \frac{c}{1 + c\|\nabla u\|^2}\nabla^2 u(E_1, E_2)\tilde{\nabla}_{\nabla u}E_1 \\ &= \nabla_{\nabla_{E_1}E_2}E_1 + \frac{c}{1 + c\|\nabla u\|^2}\nabla^2 u(E_1, E_2)\nabla_{\nabla u}E_1 + f_1\nabla u, \end{aligned}$$

and

$$(30) \quad \begin{aligned} \tilde{\nabla}_{\tilde{\nabla}_{E_2}E_1}E_1 &= \tilde{\nabla}_{\nabla_{E_2}E_1}E_1 + \frac{c}{1 + c\|\nabla u\|^2}\nabla^2 u(E_1, E_2)\tilde{\nabla}_{\nabla u}E_1 \\ &= \nabla_{\nabla_{E_2}E_1}E_1 + \frac{c}{1 + c\|\nabla u\|^2}\nabla^2 u(E_1, E_2)\nabla_{\nabla u}E_1 + f_2\nabla u, \end{aligned}$$

where  $f_1, f_2 \in \mathcal{C}^\infty(\Sigma)$ . Observe that, in order to obtain  $\tilde{K}$ , we will have to compute the product of the expressions above times  $E_2$ , which is by assumption orthogonal to  $\nabla u$ . Therefore, all the terms that are proportional to  $\nabla u$  will vanish, and so we do not mind the explicit expressions for  $f_1$  and  $f_2$ . From (29) and (30) we get

$$(31) \quad \tilde{\nabla}_{[E_1, E_2]}E_1 = \nabla_{[E_1, E_2]}E_1 + f_3\nabla u,$$

being  $f_3 = f_1 - f_2 \in \mathcal{C}^\infty(\Sigma)$ . On the other hand,

$$\begin{aligned} \tilde{\nabla}_{E_1}\tilde{\nabla}_{E_2}E_1 &= \tilde{\nabla}_{E_1}\nabla_{E_2}E_1 + \frac{c}{1 + c\|\nabla u\|^2}\nabla^2 u(E_1, E_2)\tilde{\nabla}_{E_1}\nabla u + f_4\nabla u \\ &= \nabla_{E_1}\nabla_{E_2}E_1 + \frac{c}{1 + c\|\nabla u\|^2}\nabla^2 u(E_1, E_2)\nabla_{E_1}\nabla u + f_5\nabla u, \end{aligned}$$

and

$$\begin{aligned}\tilde{\nabla}_{E_2} \tilde{\nabla}_{E_1} E_1 &= \tilde{\nabla}_{E_2} \nabla_{E_1} E_1 + \frac{c}{1 + c\|\nabla u\|^2} \nabla^2 u(E_1, E_1) \tilde{\nabla}_{E_2} \nabla u + f_6 \nabla u \\ &= \nabla_{E_2} \nabla_{E_1} E_1 + \frac{c}{1 + c\|\nabla u\|^2} \nabla^2 u(E_1, E_1) \nabla_{E_2} \nabla u + f_7 \nabla u,\end{aligned}$$

where again  $f_4, f_5, f_6, f_7 \in \mathcal{C}^\infty(\Sigma)$ . Therefore,

$$\begin{aligned}(32) \quad [\tilde{\nabla}_{E_1}, \tilde{\nabla}_{E_2}] E_1 &= [\nabla_{E_1}, \nabla_{E_2}] E_1 \\ &\quad + \frac{c}{1 + c\|\nabla u\|^2} (\nabla^2 u(E_1, E_2) \nabla_{E_1} \nabla u - \nabla^2 u(E_1, E_1) \nabla_{E_2} \nabla u) + f_8 \nabla u\end{aligned}$$

being  $f_8 = f_5 - f_7 \in \mathcal{C}^\infty(\Sigma)$ , which jointly with (31) yields

$$\begin{aligned}\tilde{R}(E_1, E_2) E_1 &= R(E_1, E_2) E_1 \\ &\quad + \frac{c}{1 + c\|\nabla u\|^2} (\nabla^2 u(E_1, E_1) \nabla_{E_2} \nabla u - \nabla^2 u(E_1, E_2) \nabla_{E_1} \nabla u) + f \nabla u\end{aligned}$$

being  $f = f_3 - f_8 \in \mathcal{C}^\infty(\Sigma)$ . Therefore,

$$\begin{aligned}(33) \quad \tilde{g}(\tilde{R}(E_1, E_2) E_1, E_2) &= g(\tilde{R}(E_1, E_2) E_1, E_2) \\ &= g(R(E_1, E_2) E_1, E_2) + \frac{c}{1 + c\|\nabla u\|^2} \det \nabla^2 u.\end{aligned}$$

Or equivalently, from (27) and (28)

$$\tilde{K} = \frac{K(1 + c\|\nabla u\|^2) + c \det \nabla^2 u}{(1 + c\|\nabla u\|^2)^2},$$

which proves the result.  $\square$

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